LESSON 13 - STUDY GUIDE

ABSTRACT. In this lesson we finally start looking at Fourier series. Following Katznelson [5, 6] we will stay within the one-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ context, beginning with the basic definitions and looking at the elementary properties of the Fourier coefficients.

1. Fourier series: definitions and basic properties of Fourier coefficients.

Study material: With the beginning of our study of Fourier series, I will follow Katznelson's book very closely, as it is considered one of the classical textbooks at this level. The older second edition [5], published by Dover in 1976, is widely popular, but a more recent third edition [6] has been published in 2004 by Cambridge University Press. For the most part they do not differ, except for minor corrections and a few added sections in the latest edition. So I will cite both, and make reference to the corresponding pages, whenever needed. For this lesson we will cover section **1 - Fourier Coefficients** from chapter **I - Fourier Series on** $\mathbb{T}$, corresponding to pgs. 1–6 in both the second and third editions [5] and [6].

Other extremely good textbooks, that cover this material at roughly the same level, but not necessarily following the exact order of presentation of topics as in Katznelson, are [1, 3, 4, 7, 8]. Finally, of course, the treatise on trigonometric series by the indisputable master and creator of the modern school of harmonic analysis of the second half of the twentieth century, Antoni Zygmund's "*Trigonometric Series*" [9], although very unreadable as a textbook, nevertheless contains literally everything known on the subject at the time that it was written, in 1959, and is still an authoritative reference, full of interesting results and information, often not found anywhere else.

In basic calculus or differential equations courses, one introduces the concept of Fourier series usually by recalling Fourier's approach to solving the heat equation by using the method of separation of variables. When trying to adjust the general solution obtained this way to the initial data, one is then led to the problem of representing a general function f as a series of sines and/or cosines, of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

on the interval $x \in [-L, L]$. After what is usually just a heuristic argument, involving linear algebra analogies with projections of vectors on orthogonal bases, students are motivated to accept that the appropriate coefficients for such a representation are necessarily the so called Fourier coefficients of f, given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \dots \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \dots$$

for f sufficiently well behaved with respect to integration, which at that level of study is typically only the Riemann definition. Convergence theorems are normally only stated, but left unproven, with at least Dirichlet's result for the pointwise convergence of Fourier series for seccionally C^1 functions being the default result presented in most elementary courses.

Date: May 4, 2020.

Of course the issue of convergence of Fourier series, or rather, the issues of convergence - as there are many different types and modes of convergence - are very deep and difficult mathematical problems. As I tried to briefly convey, when I summarized the history of harmonic analysis in the first lecture for this course, the study of Fourier series and their convergence was one of the main driving forces in the modern development of mathematical analysis, since the beginning of the nineteenth century, after Fourier's revolutionary idea. Topics such as Cantor's theory of sets, or the theories of integration, developed by Riemann and Lebesgue, arose from the increasing need to create a more rigorous and powerful mathematical structure with which to study the very subtle problems raised by the convergence of Fourier series. In particular, no modern exposition of the theory of Fourier series can proceed without routinely using Lebesgue integration and the accompanying theory of L^p spaces, because the subject depends crucially on operations involving limits and integrals that cannot suitably be performed within the context of the Riemann integral. And that is why it is so difficult to do a proper presentation at the level of calculus courses.

So we now start by establishing the mathematical context in which the study of trigonometric series will be held. In the first place, we normalize the domain by taking $L = \pi$, so that the series becomes

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx). \tag{1.1}$$

At this point this series should only be considered formally, as we are not making any assumptions concerning its convergence. Using the identities

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2} \qquad \text{and} \qquad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i},$$

we can also rewrite (1.1) as

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{1.2}$$

where the coefficients c_n are related to the a_n and b_n by

$$a_n = (c_n + c_{-n}) \qquad \text{and} \qquad b_n = i(c_n - c_{-n}) \qquad n \geq 0. \tag{1.3}$$

When written in the form (1.1), the trigonometric series is usually said to be in the *real form* while the corresponding (1.2) is known as the *complex form*. The reason being, as we will see shortly, that for real functions f the corresponding Fourier coefficients a_n and b_n are real, while the c_n are generally complex. We will work, almost exclusively, with the latter.

The trigonometric functions used in these formal series are all periodic, with fundamental periods $2\pi/n$ for each n, but with a common period of 2π among all of them. So, any function that we can expect to represent by such a series, in particular in the elementary case where it reduces to a finite sum, should necessarily also be periodic with period 2π. Therefore, in the study of trigonometric series, only complex functions defined on the real line, $f : \mathbb{R} \to \mathbb{C}$, which are periodic with period 2π, are considered.

If we identify the points on the real line whose distance to each other is an integer multiple of the period 2π, this process corresponds to taking the quotient $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ as the domain of the functions under consideration. This quotient is the set of equivalence classes defined by the equivalence relation

$$x \sim y \qquad \text{if} \qquad x - y = 2\pi k, \text{ with } k \in \mathbb{Z}.$$

More precisely, there is a natural one to one correspondence between the periodic functions defined on $\mathbb{R}$, with period 2π, and the functions defined on $\mathbb{T}$

$$f : \mathbb{R} \to \mathbb{C} \qquad \leftrightarrow \qquad \tilde{f} : \mathbb{T} \to \mathbb{C},$$

with

$$f(x) = \tilde{f}([x]),$$

where $[x]$ denotes the equivalence class of $x \in \mathbb{R}$, i.e. $[x] = \{x + 2k\pi : k \in \mathbb{Z}\}$. We will, however, not continue making this careful distinction, and will instead, from now on, regard both, the periodic function defined on $\mathbb{R}$, as well as its counterpart defined on $\mathbb{T}$, as the same object.

Addition can perfectly well be defined on $\mathbb{T}$, as the equivalence class of the resulting sum of any representatives of two equivalence classes does not depend on the choice of representatives:

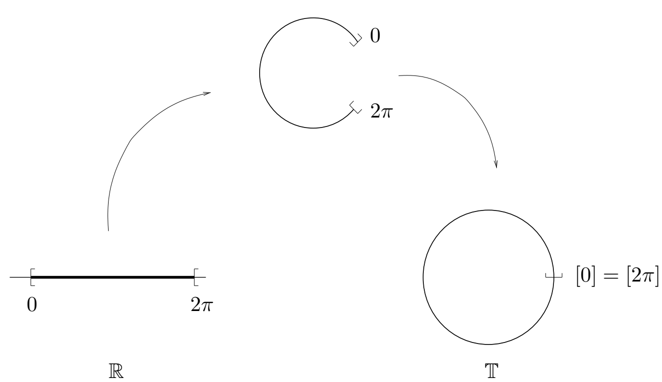
$$\text{If} \quad [x] = [x'] \quad \text{and} \quad [y] = [y'] \quad \Rightarrow \quad x = x' + 2n\pi \quad \text{and} \quad y = y' + 2m\pi$$
$$\Rightarrow \quad x + y = x' + y' + 2(n+m)\pi \quad \Rightarrow \quad [x+y] = [x' + y'].$$

This, of course, is just an algebraic consequence of performing the quotient of the additive group of real numbers $(\mathbb{R}, +)$ by the subgroup $2\pi\mathbb{Z}$. Thus, we define $[x] + [y] = [x + y]$, as well as the product of an element of $\mathbb{T}$ by an integer, $n[x] = [nx]$. The product of any two elements of $\mathbb{T}$ cannot be defined, though, as this invariance with respect to arbitrary choices of representatives does not hold for multiplication.

The natural topology that one defines on $\mathbb{T}$ is the quotient topology, where $O \subset \mathbb{T}$ is open if $\bigcup_{[x]\in O}\{x + 2k\pi \in \mathbb{R} : k \in \mathbb{Z}\}$ is open in $\mathbb{R}$, i.e., if $\pi^{-1}(O) \subset \mathbb{R}$ is open, with $\pi : \mathbb{R} \to \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denoting the canonical projection $\pi(x) = [x]$. Equipped with this quotient topology, $\mathbb{T}$ is a compact space and the binary operation of addition $+ : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is continuous, besides being commutative. Therefore, we conclude that $(\mathbb{T}, +)$ is a locally compact - actually globally compact - abelian group. It is called the *one-dimensional torus*. And, as pointed out before, the analysis of functions defined on locally compact abelian groups is precisely the central subject of harmonic analysis, from an abstract point of view. The study of Fourier series for functions on the torus $\mathbb{T}$ will be our concrete model, in this respect.

When restricted to any open-closed interval in $\mathbb{R}$ of length 2π, of the type $[a, a + 2\pi[$, for fixed $a \in \mathbb{R}$, the projection mapping $x \in [a, a + 2\pi[\mapsto \pi(x) = [x] \in \mathbb{T}$ is bijective so we can, and will, conveniently use it to identify $\mathbb{T}$ with $[0, 2\pi[$ (or, less frequently, with $] - \pi, \pi]$ also). The sum $[x] + [y]$ in $\mathbb{T}$ corresponds then to the real sum $x + y \mod 2\pi$. A set $O \in [0, 2\pi[$ corresponds to an open set in $\mathbb{T}$ if and only if the union of its 2π-periodic copies is open in $\mathbb{R}$. So, for example, the interval $]\pi, 2\pi[$ or the set $[0, \pi/2[\cup]3\pi/2, 2\pi[=] - \pi/2, \pi/2[$ are open in $\mathbb{T}$, but the interval $[0, \pi[$ is not. Of course, the full set $[0, 2\pi[$ is, in this topology, both open, closed and compact.

Geometrically one can imagine $\mathbb{T}$ as the result of curving the interval $[0, 2\pi[$ on $\mathbb{R}$ into a circle, with the extreme points 0 and 2π glued together.



In fact, this geometric identification between the torus $\mathbb{R}/2\pi\mathbb{Z}$ and the closed circle can also be rigorously made through the isomorphism between compact abelian groups $e^{it} : (\mathbb{T}, +) \to (\mathbb{S}^1, \times)$ where $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane with the abelian group operation being the usual complex multiplication. This group isomorphism is so significant that it is more frequent to call $\mathbb{T}$ the unit circle than one-dimensional torus although, on the other hand, one tends more often to think about it as the interval $[0, 2\pi[$ in the additive group of the real numbers, than as $\mathbb{S}^1$ in the multiplicative group of the complexes.

The remaining ingredient of harmonic analysis on locally compact abelian groups, that we still have not mentioned, is the Haar measure: it is the invariant measure with respect to the group operation, which can be proved to always exist and is unique, modulo a positive multiplicative constant (see Folland's book [2], for example). Of course, on $(\mathbb{T}, +)$, the Haar measure is any positive constant multiple of the Lebesgue measure, corresponding to the restriction of the Lebesgue measure on $\mathbb{R}$ to the interval $[0, 2\pi[$. We will still denote it in the usual way by dt, and therefore an integrable function on $\mathbb{T}$, $f \in L^1(\mathbb{T})$, simply corresponds to a periodic function on $\mathbb{R}$, with period 2π, which is integrable on the interval $[0, 2\pi[$ (or on any other interval of length 2π, for that matter). And we have

$$\int_{\mathbb{T}} f(t)dt = \int_0^{2\pi} f(t)dt = \int_{-\pi}^{\pi} f(t)dt,$$

where, to be completely rigorous, the function and the measure on the left hand side of the previous identities are defined on $\mathbb{T}$, while the function on the middle and right hand sides is the 2π-periodic counterpart defined on $\mathbb{R}$ and the measure is the Lebesgue measure restricted to the intervals $[0, 2\pi[$ and $]-\pi, \pi]$. The total measure of $\mathbb{T}$ is then finite, a consequence of the group being compact, and equals 2π if we take the Haar measure to be exactly equal to the Lebesgue measure. We will see, as we proceed with the study of the properties of Fourier series, that some formulas would become simpler if we normalized the measure to be unitary on the whole of $\mathbb{T}$, i.e., by considering a new measure $dx = dt/2\pi$. However, dividing the Lebesgue measure at the outset by 2π tends to be confusing, specially when dealing with the length of subintervals that we always tend to think about as in $\mathbb{R}$. So it is instead more frequent to keep the Lebesgue measure and explicitly divide it by 2π on every formula. Another common alternative is to start by considering the one-dimensional torus not as $\mathbb{R}/2\pi\mathbb{Z}$, but as $\mathbb{R}/\mathbb{Z} \simeq [0, 1[$, with the corresponding periodic functions having period one. But then the price is paid in the exponentials of the trigonometric series, that will always have to carry the 2π factor in their exponents. The invariance of the Haar measure with respect to the group operation corresponds, in our case, to the invariance of the Lebesgue integral on $\mathbb{T}$ with respect to the translation of integrable functions

$$\begin{aligned}\int_{\mathbb{T}} f(t)dt = \int_0^{2\pi} f(t)dt = \int_{t_0}^{t_0+2\pi} f(t)dt = \\ = \int_0^{2\pi} f(t-t_0)dt = \int_{\mathbb{T}} f(t-t_0)dt = \int_{\mathbb{T}} \big(\tau_{t_0} f\big)(t)dt, \quad \text{for any} \quad t_0 \in \mathbb{T}.\end{aligned}$$

Note that the concepts of continuity, measurability and differentiability of functions on $\mathbb{T}$, with respect to its topological and measure space structure, are exactly equivalent to the usual concepts for their 2π-periodic counterparts on $\mathbb{R}$. And again, we will not distinguish these definitions for each of the two forms of looking at the functions, constantly interchanging between one or the other, depending on which one is more convenient for any particular situation.

To finish this mathematical setup, we define the $L^p(\mathbb{T})$ norms of measurable functions on $\mathbb{T}$ with the measure normalized (explicitly) by the 2π factor, so that

$$\|f\|_{L^p(\mathbb{T})} = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|^p dt\right)^{\frac{1}{p}}, \quad \text{for} \quad 1 \leq p < \infty,$$

while

$$\|f\|_{L^\infty(\mathbb{T})} = \operatorname*{ess\,sup}_{t\in\mathbb{T}} |f(t)|.$$

Recall that, from the inclusion properties of L^p spaces, as $\mathbb{T}$ has finite measure, we have

$$L^\infty(\mathbb{T}) \subset L^p(\mathbb{T}) \subset L^1(\mathbb{T}), \quad \text{for} \quad 1 \leq p < \infty,$$

and, because of the normalization of the measure, we have the inequalities with unit constants

$$\|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^p(\mathbb{T})} \leq \|f\|_{L^\infty(\mathbb{T})}.$$

As always, we consider the $L^p(\mathbb{T})$ as the set of classes of equivalence of functions almost everywhere equal, for which the corresponding $L^p(\mathbb{T})$ norm is finite.

We can, at last, start presenting the first definitions related to Fourier series. So, if we consider the partial sums associated with the series (1.1) or (1.2) given, respectively by

$$\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nt) + b_n \sin(nt), \tag{1.4}$$

and the equivalent symmetric partial sum

$$\sum_{n=-N}^{N} c_n e^{int}, \tag{1.5}$$

with arbitrary complex coefficients a_n, b_n and c_n related by (1.3), then, because these sums are finite, they define infinitely differentiable functions on $\mathbb{T}$. They are called *trigonometric polynomials.*

Definition 1.1. *A function $P : \mathbb{T} \to \mathbb{C}$ defined by a finite sum of the form* (1.4) *or* (1.5)*, with $a_n, b_n, c_n \in \mathbb{C}$ such that c_N, c_{-N}, a_N or b_N are not zero, is called a trigonometric polynomial of degree N. The integers n are called the* frequencies *of the polynomial, and the corresponding coefficients $a_n, b_n, c_n \in \mathbb{C}$ are sometimes also called the* amplitudes of the oscillations*, at frequency n.*

As the sines, cosines and exponentials satisfy the orthogonality relations

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)\sin(mt)dt = 0, \quad \text{for all} \quad n \geq 0, m \geq 1,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \end{cases}, \quad \text{for all} \quad n, m \geq 0,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \end{cases}, \quad \text{for all} \quad n, m \geq 1,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}e^{-imt}dt = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \end{cases}, \quad \text{for all} \quad n, m \in \mathbb{Z},$$

then we can easily conclude that the trigonometric polynomial P uniquely defines its coefficients a_n, b_n and c_n. In fact, multiplying P by sines, cosines or exponentials, and integrating over $\mathbb{T}$ the previous orthogonality formulas yield

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nt)P(t)dt, \quad 0 \leq n \leq N \tag{1.6}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nt)P(t)dt, \quad 1 \leq n \leq N \tag{1.7}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}P(t)dt, \quad -N \leq n \leq N. \tag{1.8}$$

Due to these formulas, the coefficients are thus also called the *Fourier coefficients* of the trigonometric polynomial P. Observe that, if $P : \mathbb{T} \to \mathbb{R}$ is a real function, then the Fourier coefficients a_n, b_n are real, while that is not generally the case with the c_n, which are complex even for real P. This is the reason why (1.1) and (1.4) are called the real forms of the, respectively, trigonometric series and trigonometric polynomial, while (1.2) and (1.5) are called the complex forms.

Passing from finite to infinite sums, we naturally have the following definition.

Definition 1.2. *Let $\{c_n\}_{n\in\mathbb{Z}} \in \mathbb{C}$ be any complex sequence. Then, the* trigonometric series *associated to this sequence is the (formal) series*

$$S \sim \sum_{n=-\infty}^{\infty} c_n e^{int},$$

where we make no assumptions about convergence, and that is the reason for using the symbol $\sim$. The real form of this series is given by (1.1) *with the corresponding sequences of coefficients $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 1}$ related to the sequence $\{c_n\}_{n\in\mathbb{Z}} \in \mathbb{C}$ by* (1.3)*.*

Notice however that, from not having any assurance about the convergence of a trigonometric series, now we cannot reason as above to conclude that the coefficients should be uniquely determined by the series, with integral formulas analogous to the ones yielding the Fourier coefficients of the trigonometric polynomials. In fact, even when they do converge, one of the most important problems in the study of trigonometric and Fourier series, initially studied by Riemann and Cantor, is the issue of uniqueness of representations of functions by trigonometric series. Solving this problem is not at all trivial and we will see later in the course that there are certain situations in which the representation is actually not unique.

Motivated by formula (1.8), which nevertheless makes sense for any function $f \in L^1(\mathbb{T})$, because then $|e^{-int}f(t)| \leq |f(t)| \in L^1(\mathbb{T})$, we define the Fourier coefficients of any $f \in L^1(\mathbb{T})$ by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-int}f(t)dt, \quad n \in \mathbb{Z}. \tag{1.9}$$

Definition 1.3. *Let $f \in L^1(\mathbb{T})$. Then, the* Fourier series *of f is the trigonometric series associated to the Fourier coefficients* (1.9)

$$S[f] \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}.$$

Thus, not every trigonometric series is a Fourier series: only those whose coefficients arise from the Fourier coefficients of a function $f \in L^1(\mathbb{T})$. As we will see along this course, there exist convergent trigonometric series that are not Fourier series, and there exist Fourier series that diverge at every $t \in \mathbb{T}$. The elementary fact that not every complex sequence $\{c_n\}_{n\in\mathbb{Z}} \in \mathbb{C}$ corresponds to the Fourier coefficients of a function $f \in L^1(\mathbb{T})$ is a simple consequence of the following list of basic properties of the Fourier coefficients which, obviously, not all sequences $\{c_n\}_{n\in\mathbb{Z}} \in \mathbb{C}$ satisfy.

Theorem 1.4. *Let $f, g \in L^1(\mathbb{T})$. Then*

1. $\widehat{(f+g)}(n) = \hat{f}(n) + \hat{g}(n)$ *and, for every $\alpha \in \mathbb{C}$, $\widehat{(\alpha f)}(n) = \alpha \hat{f}(n)$.*
2. $\widehat{(\tau_h f)}(n) = \widehat{\big(f(\cdot - h)\big)}(n) = e^{-ihn}\hat{f}(n)$, *for $h \in \mathbb{T}$.*
3. $\widehat{(e^{ikt} f)}(n) = \hat{f}(n-k)$.
4. *If $\bar{f}(t) = \overline{f(t)}$ denotes the conjugate function, then $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}$.*
5. *The sequence $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ is bounded and $|\hat{f}(n)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(t)|dt = \|f\|_{L^1(\mathbb{T})}$.*

Proof. The proof of all these properties is very simple and is left as an exercise. □

From this set of properties, we therefore conclude that the operator that takes $f \in L^1(\mathbb{T})$ and maps it to the complex sequence $\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ is a linear bounded operator $L^1(\mathbb{T}) \to l^\infty(\mathbb{Z})$. We call it the *Fourier transform* and denote it by $\mathcal{F}$, so that

$$\begin{aligned} \mathcal{F} : L^1(\mathbb{T}) &\to l^\infty(\mathbb{Z}) \\ f &\mapsto \mathcal{F}(f)(n) = \hat{f}(n), \end{aligned}$$

with the bound, from property (4),

$$\|\mathcal{F}(f)\|_{l^\infty(\mathbb{Z})} \le \|f\|_{L^1(\mathbb{T})}.$$

Keep in mind, though, that not even every sequence in $l^\infty(\mathbb{Z})$ is the Fourier transform of a function in $L^1(\mathbb{T})$: that is the case, for example, with the constant sequence $c_n = 1$ for all n (which, actually, is the Fourier transform of the Dirac δ distribution, or measure). So the Fourier transform *is not* surjective from $L^1(\mathbb{T})$ to $l^\infty(\mathbb{Z})$. A simple corollary of the boundedness of the Fourier transform operator is the following property.

Corollary 1.5. *If $f_j \to f$ in the $L^1(\mathbb{T})$ norm, then $\hat{f}_j(n) \to \hat{f}(n)$ uniformly.*

As we saw in the previous lessons, the convolution is a central operation in harmonic analysis, as it brings together the translation, related to underlying group operation, with integration, related to the Haar measure. We define the convolution on $\mathbb{T}$ with the 2π factor associated to the measure, as

$$f * g(t) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)g(s)ds.$$

We also saw in lesson 10, as a consequence of Young's inequality for convolutions, that if $f, g \in L^1$, then $f * g \in L^1$, with $\|f * g\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}$. Although we proved this fact for $\mathbb{R}^n$, the same fact holds for $\mathbb{T}$ exactly the same way. We will now see how simply the Fourier operates on convolutions.

Theorem 1.6. *Let $f, g \in L^1(\mathbb{T})$. Then, $f * g \in L^1(\mathbb{T})$ and, for every $n \in \mathbb{Z}$, we have*

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).$$

Proof. We already know from Young's inequality that $f * g \in L^1(\mathbb{T})$, so we can compute its Fourier coefficients. Therefore

$$\widehat{f * g}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} f * g(t)e^{-int}dt = \frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)g(s)ds\right)e^{-int}dt,$$

and using Fubini's theorem, to exchange the order of integration, as well as property (2) in Theorem 1.4, we obtain

$$\frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{1}{2\pi}\int_{\mathbb{T}} f(t-s)e^{-int}dt\right)g(s)ds = \frac{1}{2\pi}\int_{\mathbb{T}} \hat{f}(n)e^{-ins}g(s)ds = \hat{f}(n)\hat{g}(n).$$

□

Therefore, this fundamental property shows that the Fourier transform maps the convolution, which is the Banach algebra product on $L^1(\mathbb{T})$, to a product in $l^\infty(\mathbb{Z})$.

We conclude this lesson with a simple related fact, about the convolution of functions $f \in L^1(\mathbb{T})$ with trigonometric polynomials. We can think of it as resulting from Theorem 1.6, if we consider g to be a trigonometric polynomial of degree N, $P_N(t) = \sum_{-N}^{N} c_n e^{int}$, and observe that its Fourier coefficients are $\widehat{P_N}(n) = c_n$, for $-N \le n \le N$, and 0 otherwise. So that the resulting Fourier coefficients of $f * P_N$ necessarily are then $\widehat{f * P_N}(n) = c_n \hat{f}(n)$, for $-N \le n \le N$, and 0 otherwise, which basically is the same as saying that the Fourier series of $f * P_N$ is another trigonometric polynomial of degree N, with coefficients $c_n \hat{f}(n)$.

Proposition 1.7. *Let $f \in L^1(\mathbb{T})$ and $K_N(t) = \sum_{-N}^{N} c_n e^{int}$ a trigonometric polynomial of degree N. Then*

$$f * K_N(t) = \sum_{-N}^{N} c_n \hat{f}(n) e^{int}.$$

Proof. From the linearity (or distributive) property of the convolution, seen in lesson 10, we know that

$$f * K_N(t) = f * \left(\sum_{-N}^{N} c_n e^{int} \right) = \sum_{-N}^{N} c_n f * e^{int}.$$

But, denoting by φ_n the exponential at frequency n, i.e. $\varphi_n(t) = e^{int}$, we have

$$f * \varphi_n(t) = \frac{1}{2\pi} \int_{\mathbb{T}} f(t-s) \varphi_n(s) ds = \frac{1}{2\pi} \int_{\mathbb{T}} f(t-s) e^{ins} ds = \hat{f}(n) e^{int},$$

and this yields the result. □

References

[1] R. E. Edwards, *Fourier Series, A Modern Introduction - Vol. 1,* 2nd Edition, Springer-Verlag, 1979.
[2] Gerald B. Folland, *Real Analysis, Modern Techniques and Applications,* 2nd Edition, John Wiley & Sons, 1999.
[3] Loukas Grafakos, *Classical Fourier Analysis,* 3rd Edition, Springer, Graduate Texts in Mathematics 249, 2014.
[4] Henry Helson, *Classical Fourier Analysis,* Wadsworth & Brooks/Cole, 1991.
[5] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 2nd Edition, Dover Publications, 1976.
[6] Yitzhak Katznelson *An Introduction to Harmonic Analysis,* 3rd Edition, Cambridge University Press, 2004.
[7] Mark Pinsky, *Introduction to Fourier Analysis and Wavelets,* American Mathematical Society, Graduate Studies in Mathematics 102, 2009.
[8] Alberto Torchinsky, *Real-Variable Methods in Harmonic Analysis,* Dover Publications, 2004.
[9] Antoni Zygmund *Trigonometric Series,* 3rd Edition, Cambridge University Press, 2003.